Radiometer equation

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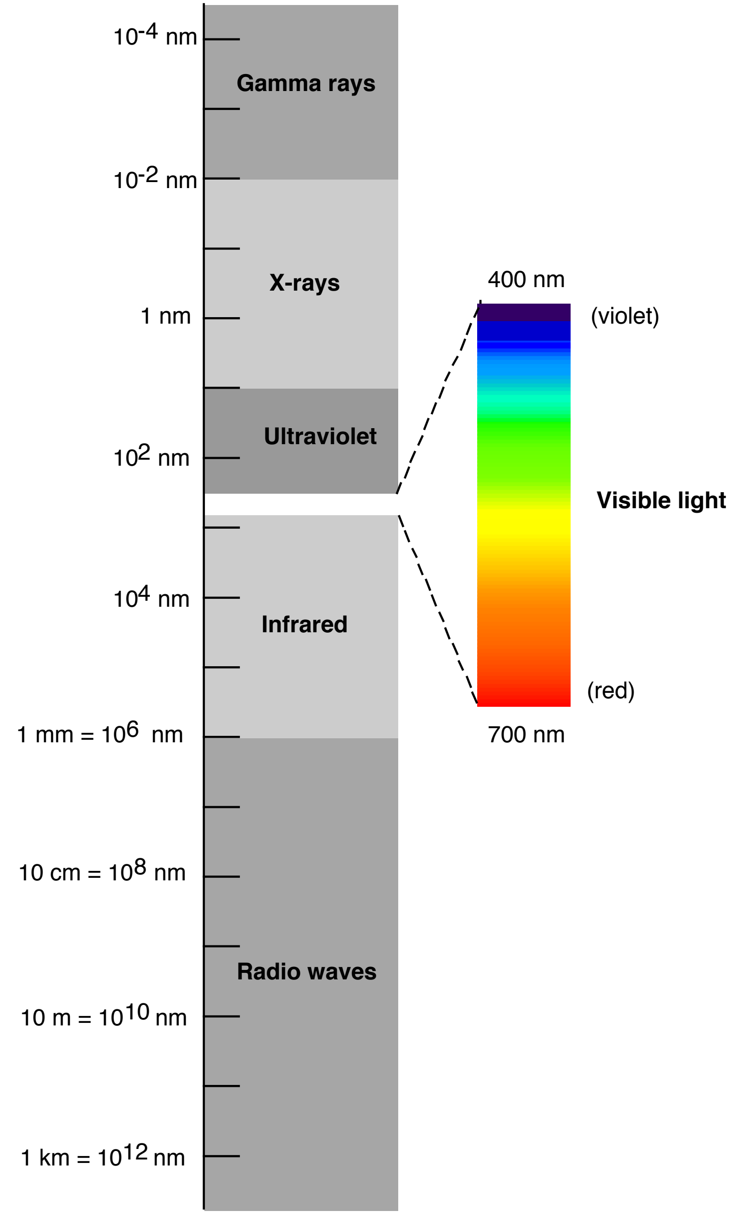
Overview

- Radio waves – how are they produced
- How do we look at radio signals
- Antenna basics (details next lecture)
- Brightness, temperature, flux density…
- Radiometer equation – sensitivity

Electromagnetic radiation

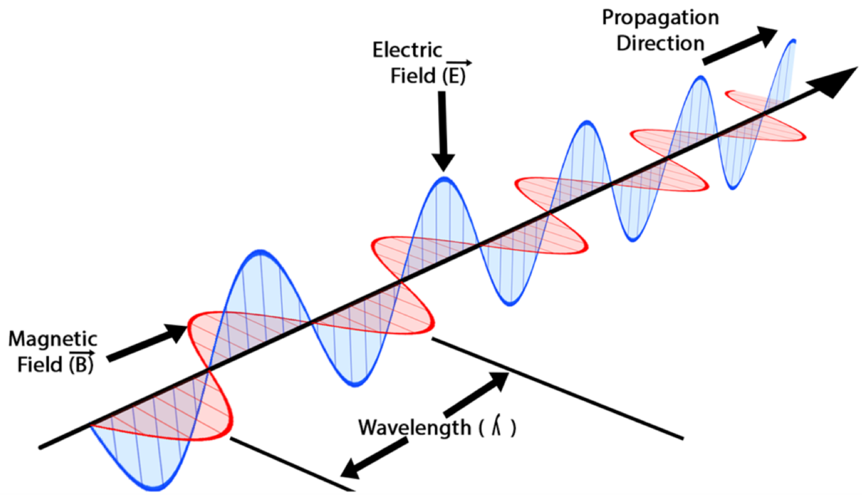
- Visible light has wavelengths 400 – 700 nm
- Radio has wavelengths longer than 1 mm (not a hard and fast definition)
- Sub-mm (0.1 – 1 mm) can also use radio techniques
- Millimetre (1 – 10 mm)
- Microwave (1 – 10s cm)
- Frequency = speed of light / wavelength
  - 10 GHz == 3 cm
  - 1 GHz == 30 cm
  - 100 MHz = 300 cm

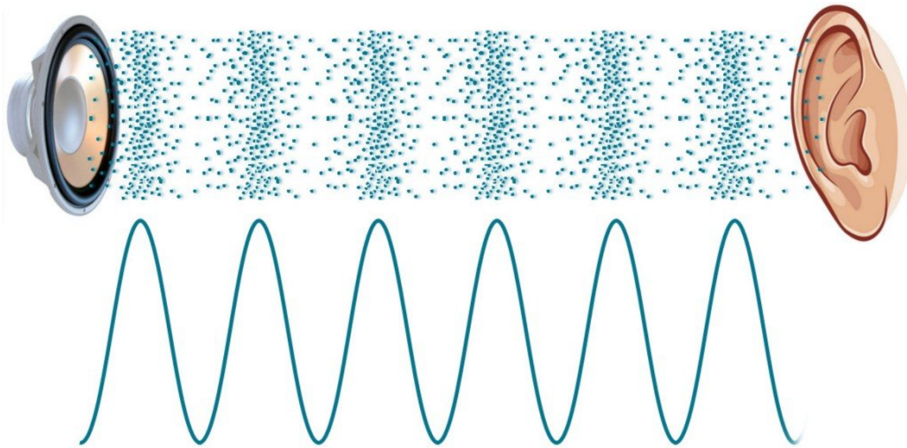


Radio vs sound

- Radio travels in a vacuum
- Radio waves are transverse
- Radio waves have polarization (direction of transverse field)

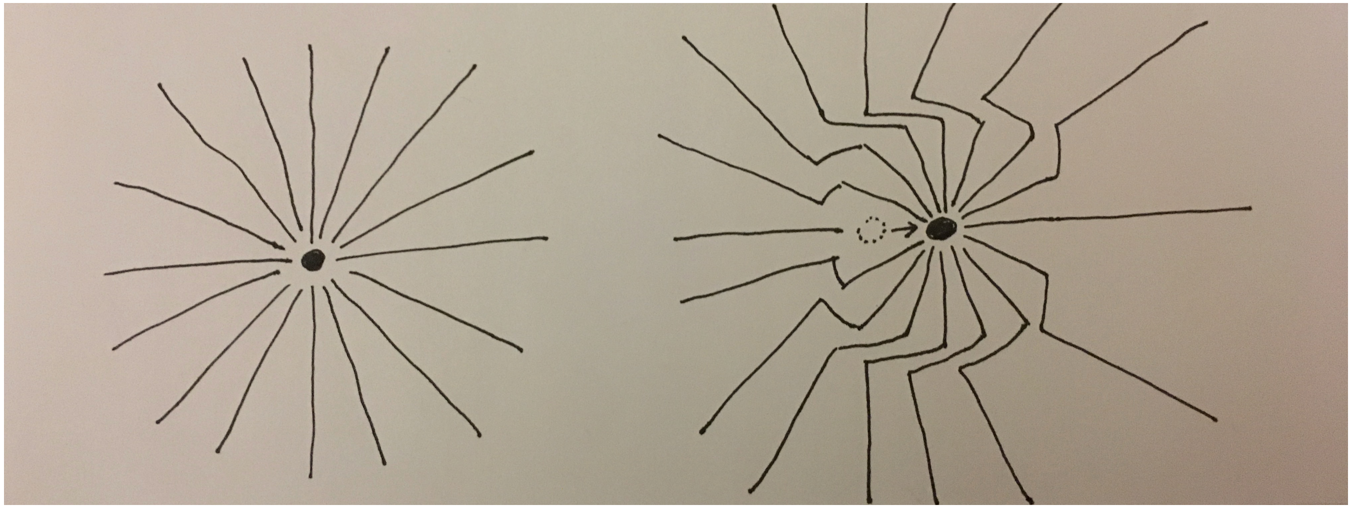
- Sound is waves in a medium (air, water, solids)
- Sound waves are longitudinal
- Sound waves not polarized





Electromagnetic radiation

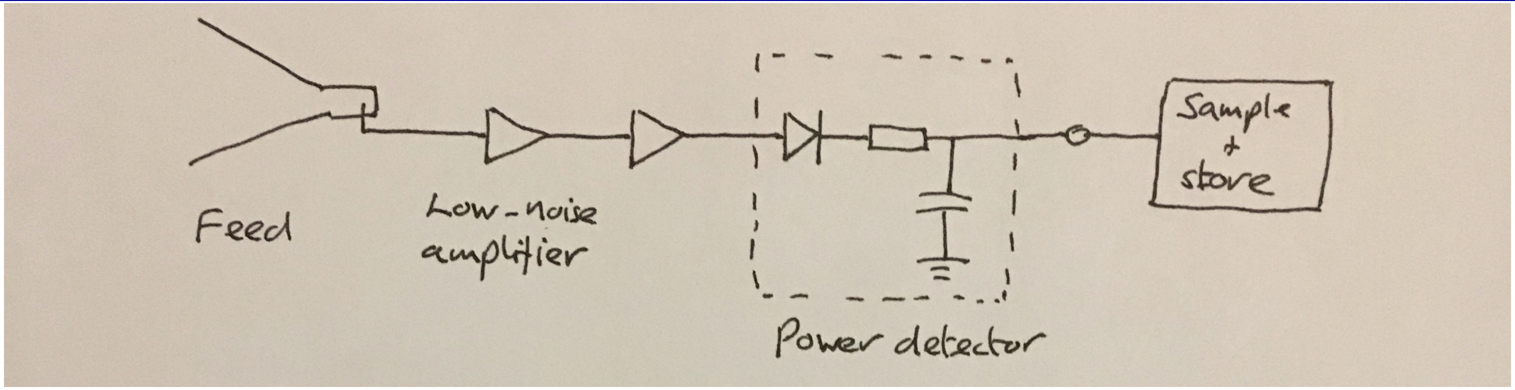
- Radio, light, X-rays, infra-red etc… are all *electromagnetic radiation*
- Described by Maxwell's equations – linking electric and magnetic fields.
- A changing electric field creates a magnetic field – and a changing magnetic field creates an electric field.
- As soon as you create a varying electric field, you generate E-M radiation.



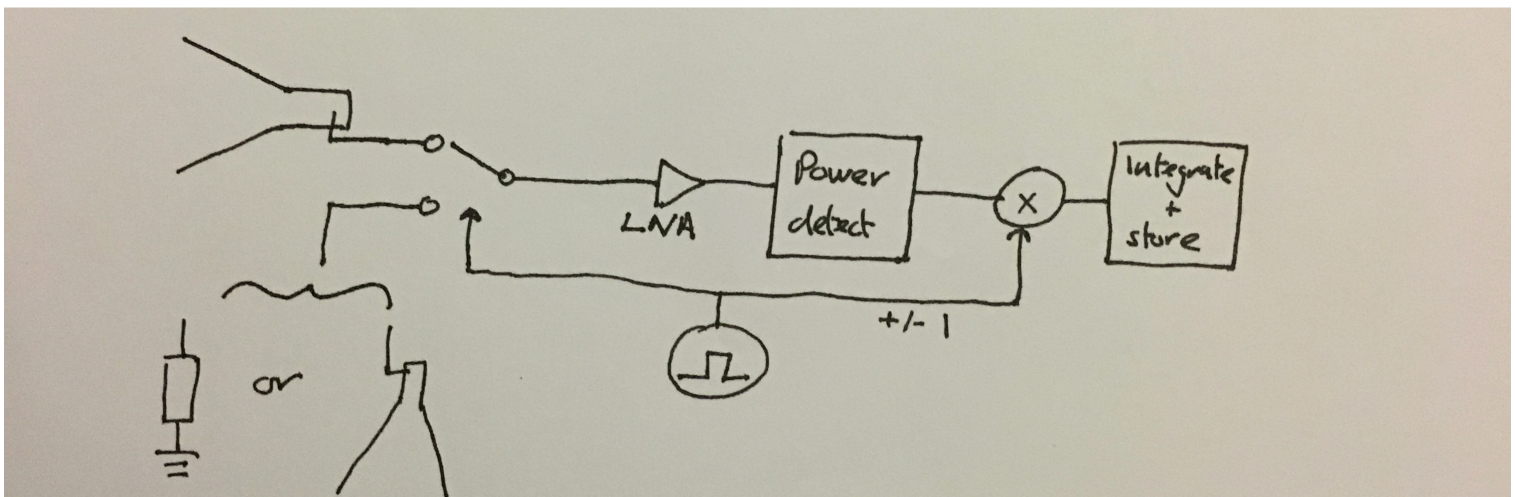
Generating and receiving radio

- To generate radio – accelerate an electric charge
- In nature –
  - Thermal motion of electrons
  - Electrons influence by electric or magnetic fields
  - Changes of bound states in atoms
- Artificially – shake electrons in a wire using a voltage
- Receiving is exactly the same process in reverse!
- The received power is proportional to the voltage squared

Radiometers



Simple radiometer – not useful in practice because amplifiers are not stable

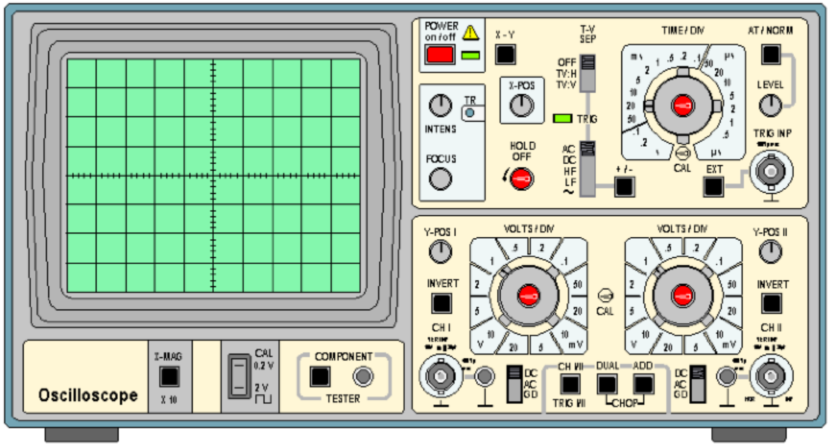


Switching radiometer – measures signal relative to a reference, through the same amplifier

Radio signal to voltage

- An antenna converts the electric field to a voltage
- We can look at the voltage either as a function of time or power as a function of frequency:

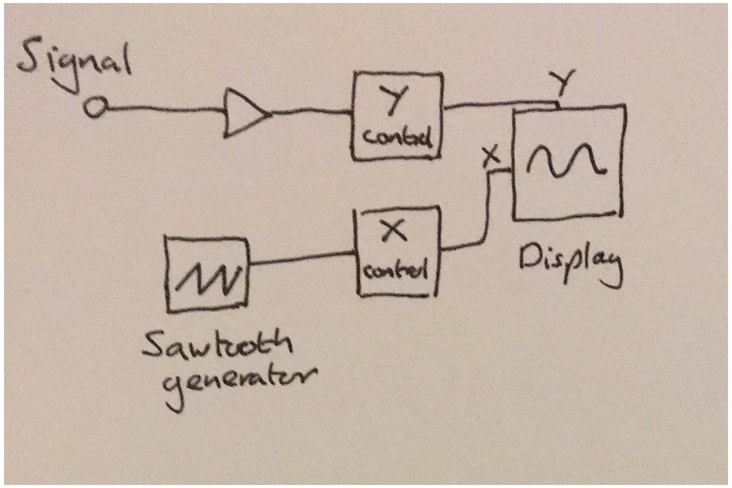
- $V(t)$: Oscilloscope



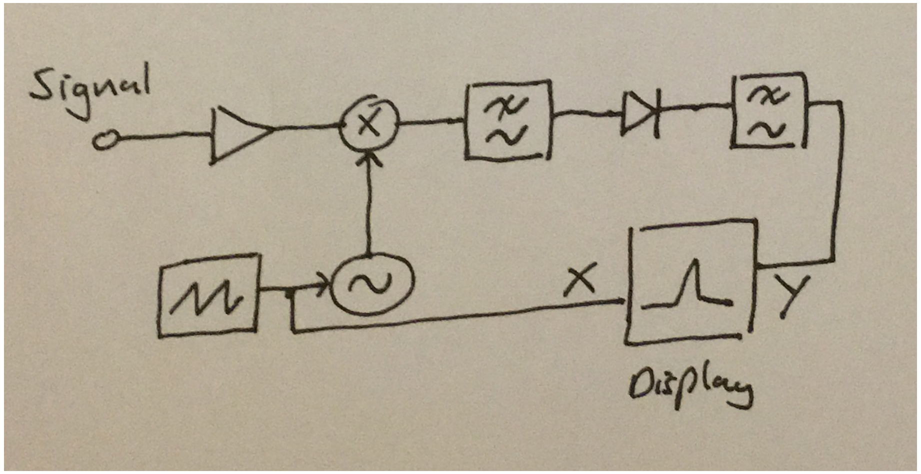
- $V(\nu)$: Spectrum analyser

What do they do?

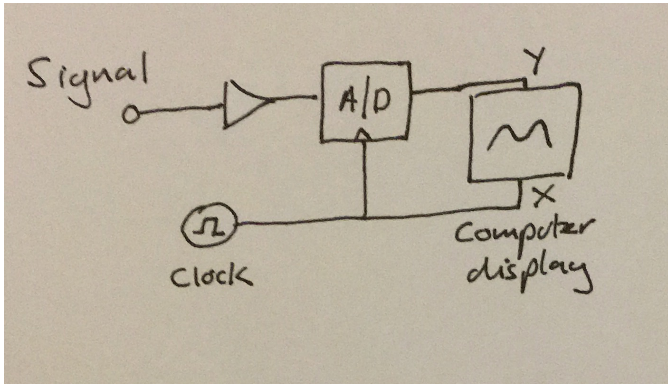
- Oscilloscope (analogue):



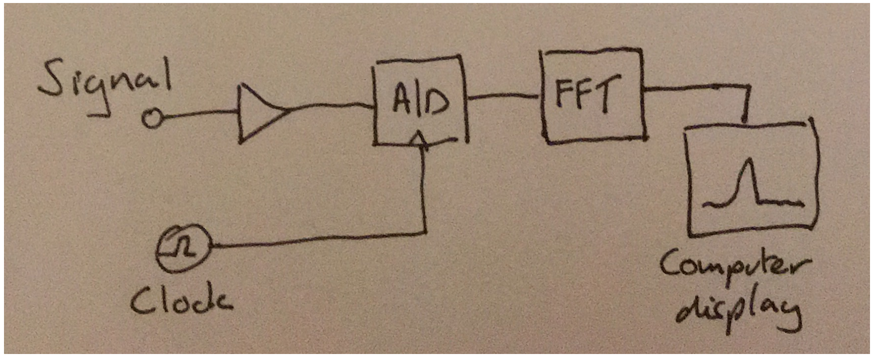
- Spectrum Analyser (analogue)



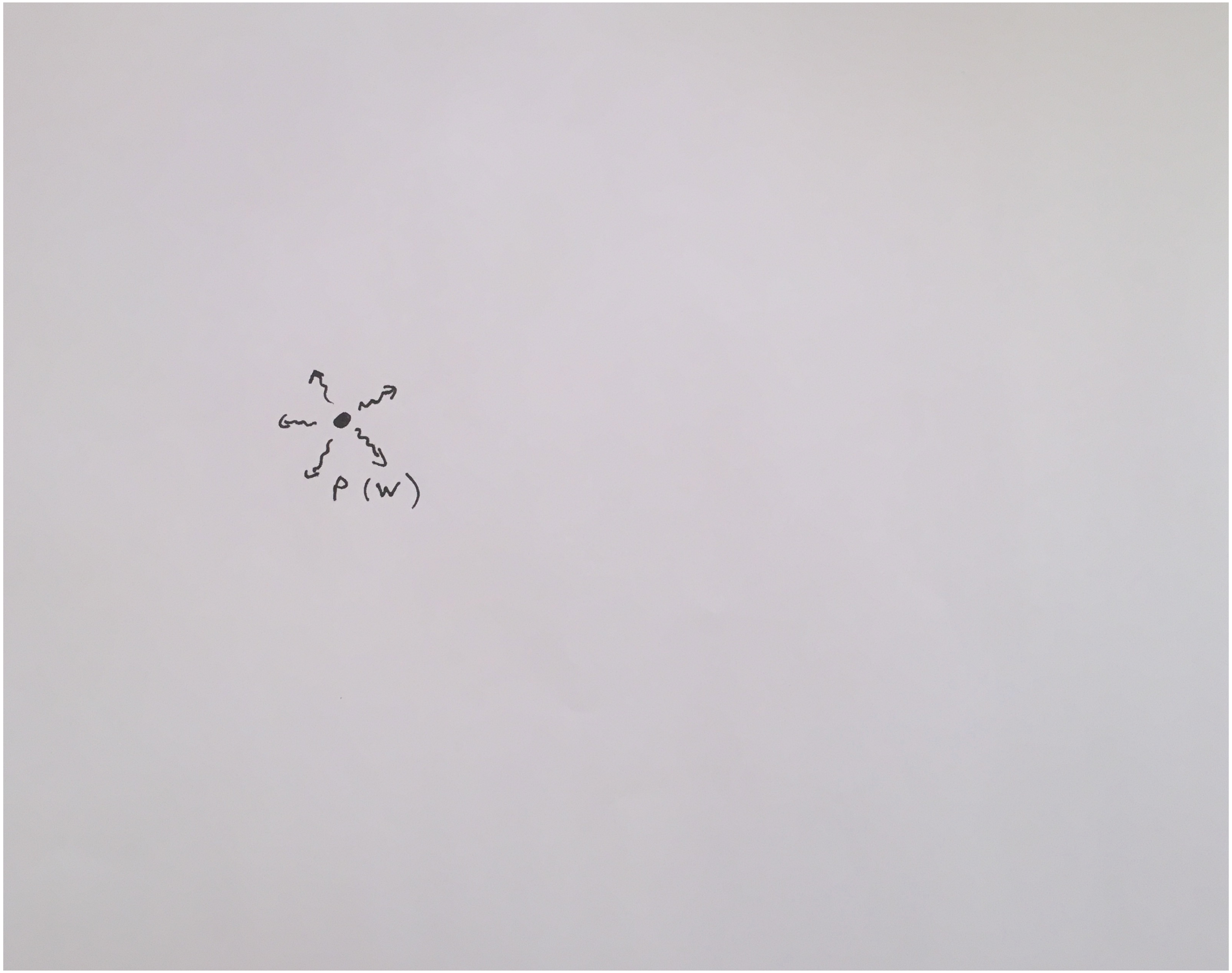
- Oscilloscope (digital)



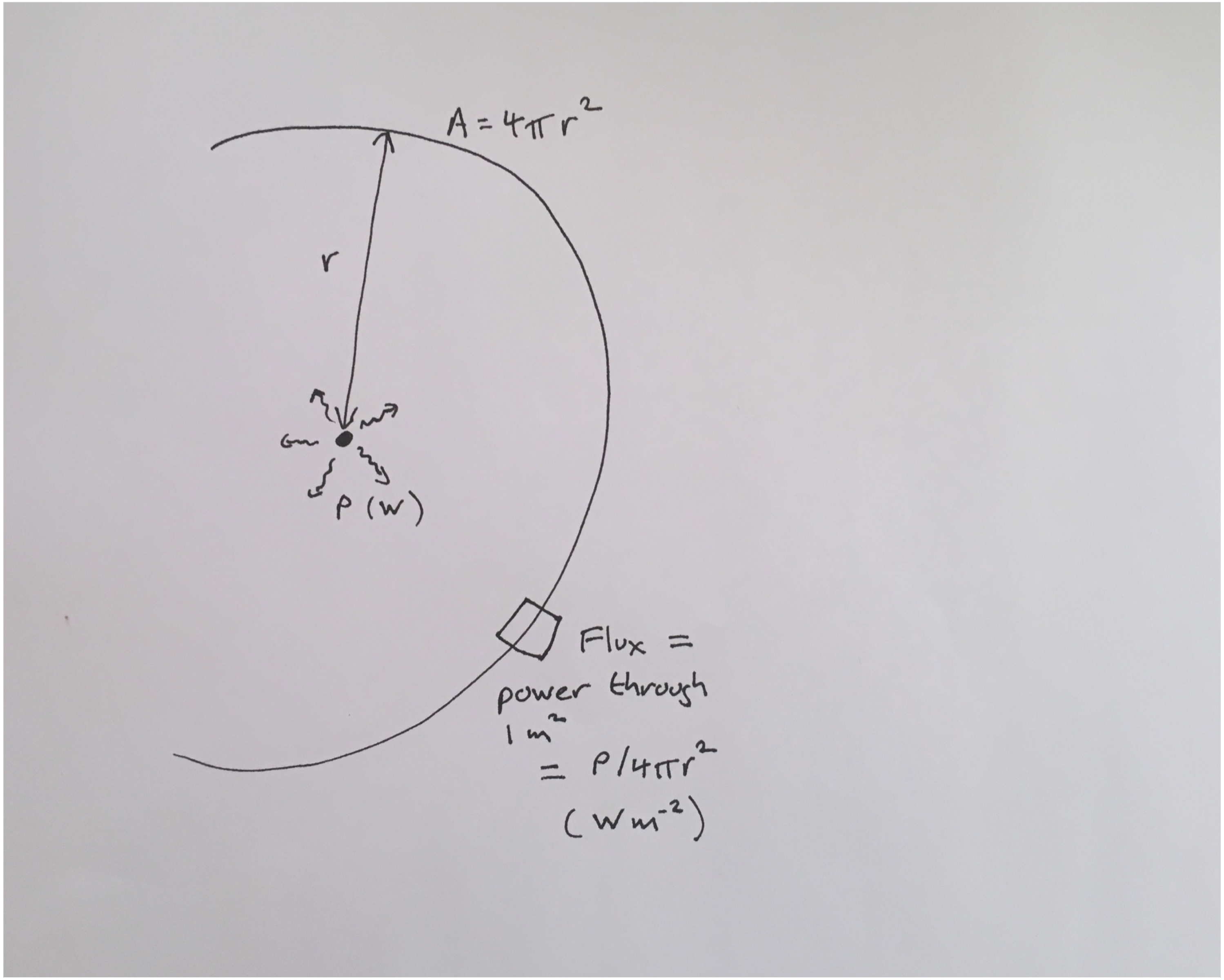
- Spectrum Analyser (digital)



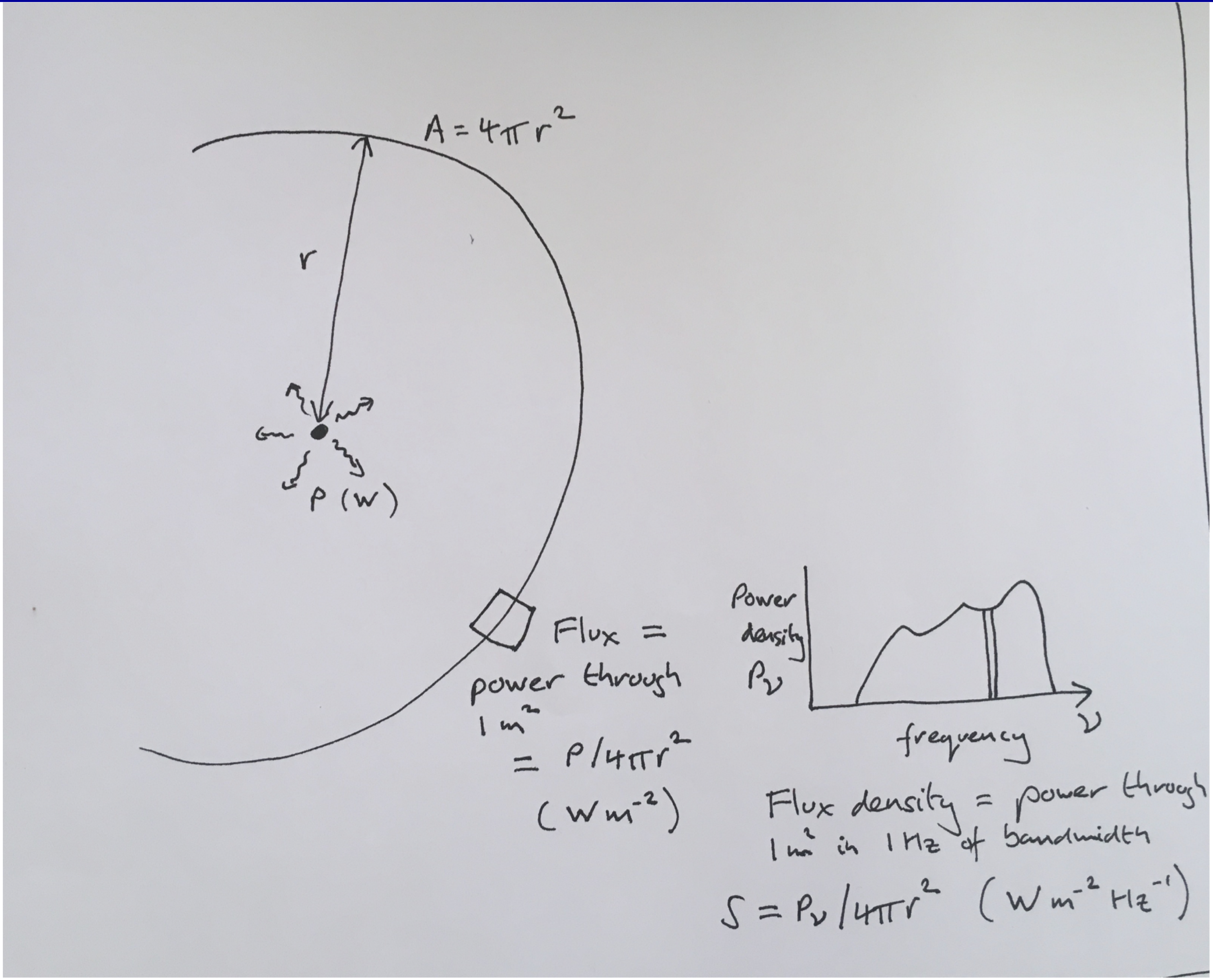
Power and flux density



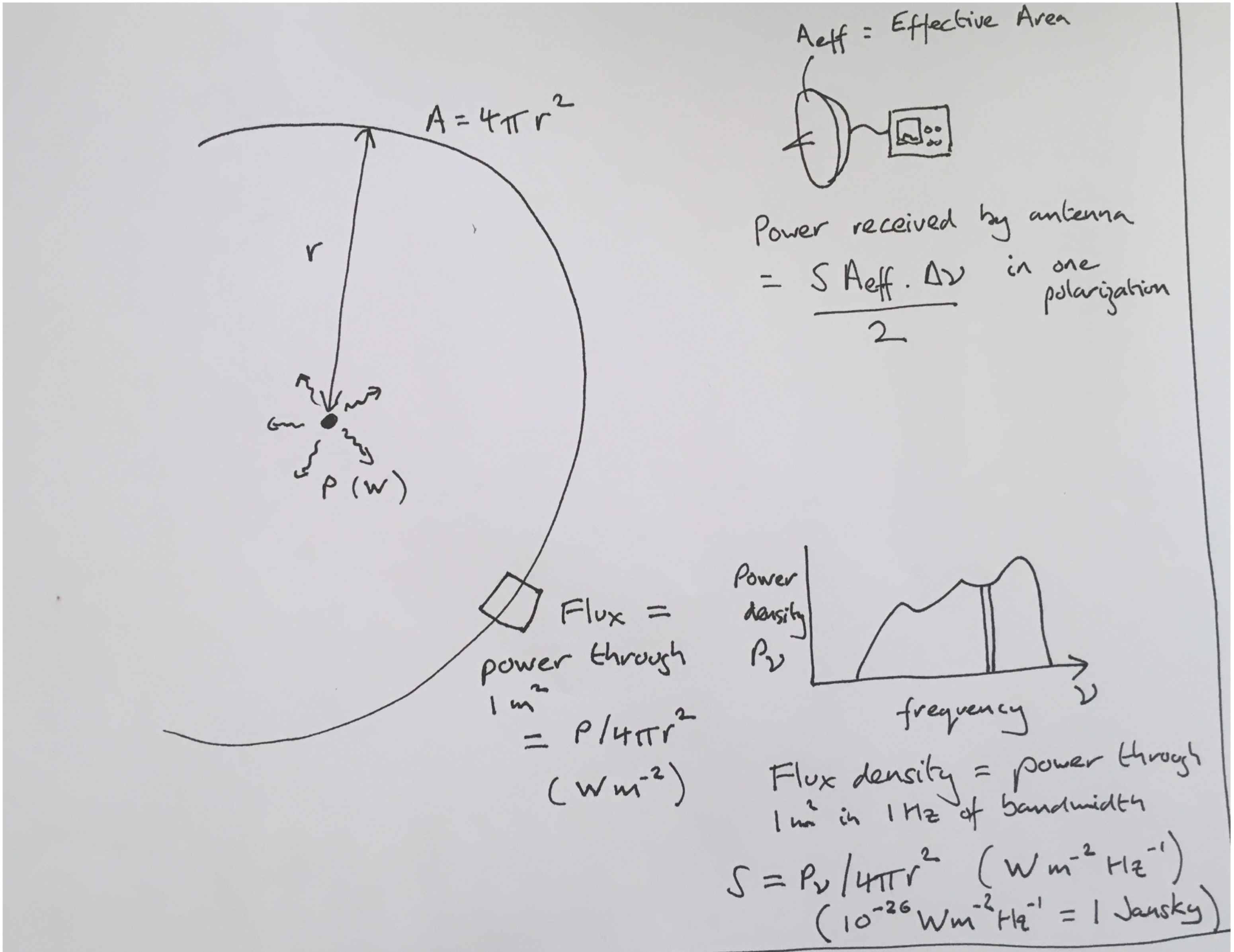
Power and flux density



Power and flux density



Power and flux density



Antenna basics

- An antenna collects radiation over an area A_{eff}
- $A_{\mathrm{eff}} = \eta A$, η is the *aperture efficiency*
- The antenna collects radiation over a solid angle Ω, where

$$\Omega A_{\mathrm{eff}} = \lambda^2$$

- An antenna that collects radiation equally from over the whole sky (*omnidirectional*) has

$$A_{\mathrm{eff}} = \lambda^2/4\pi$$

and an antenna has a maximum gain relative to omnidirectional of

$$G = 4\pi A_{\mathrm{eff}} / \lambda^2$$

Power and temperature

- A matched resistor at temperature T delivers power $kT\Delta\nu$ in to a transmission line
- So an antenna looks like a thermal load of temperature T_a, where

$$kT_a \Delta\nu = SA_{eff} \Delta\nu/2$$

$$T_a = SA_{eff}/2k$$

T_a is called the *antenna temperature*

- It's not the temperature of the antenna!
- It depends on the size of the antenna (A_{eff}). The bigger the antenna, the more temperature (K) for a given flux density (Jy)

Power and temperature

- Example: a 26-m antenna looking at the Crab Nebula at 5 GHz.

$A_{\mathrm{eff}} = \eta\pi\,(26/2)^2 = 371\ \mathrm{m}^2$ if $\eta = 0.7$

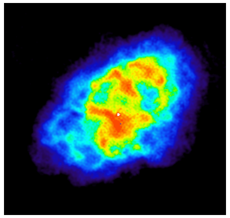
$S = 600\ \mathrm{Jy} = 600 \times 10^{-26}\ \mathrm{W\,m^{-2}\,Hz^{-1}}$

$T_{\mathrm{a}} = (6 \times 10^{-24} \times 371) / (2 \times 1.38 \times 10^{-23})$

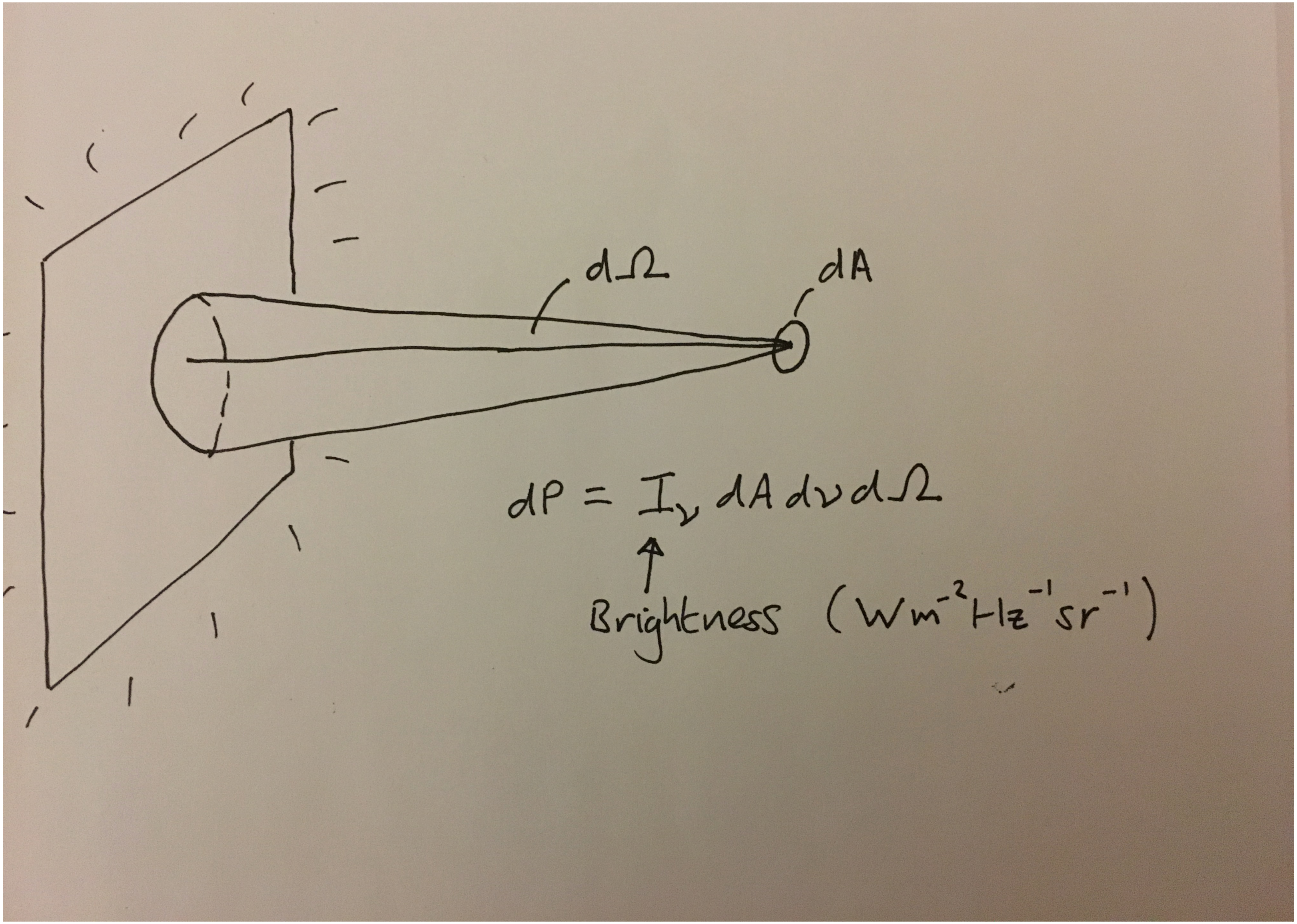
$T_{\mathrm{a}} = 80\ \mathrm{K}$

i.e. antenna gain is

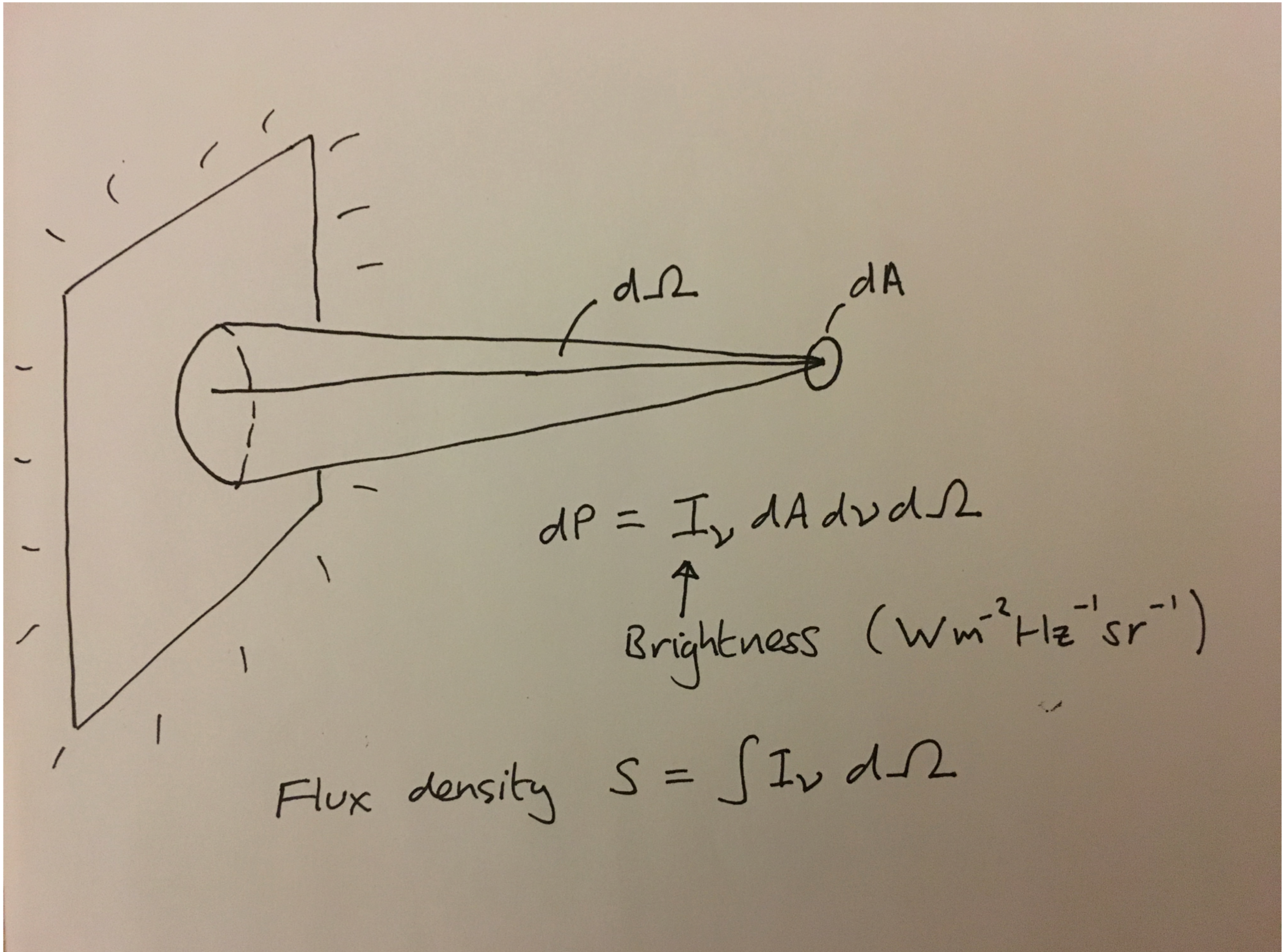
600/80 = 7.5 Jy/K



Brightness



Brightness



Brightness

$$S = \int I_\nu \, d\Omega$$

This is one of the most important definitions in radio astronomy!

Flux density is the integral of brightness over solid angle. ⟷ Brightness is flux density per solid angle.

Radio astronomers always talk about flux density and brightness (or brightness temperature…coming up).

It's easy to get confused!

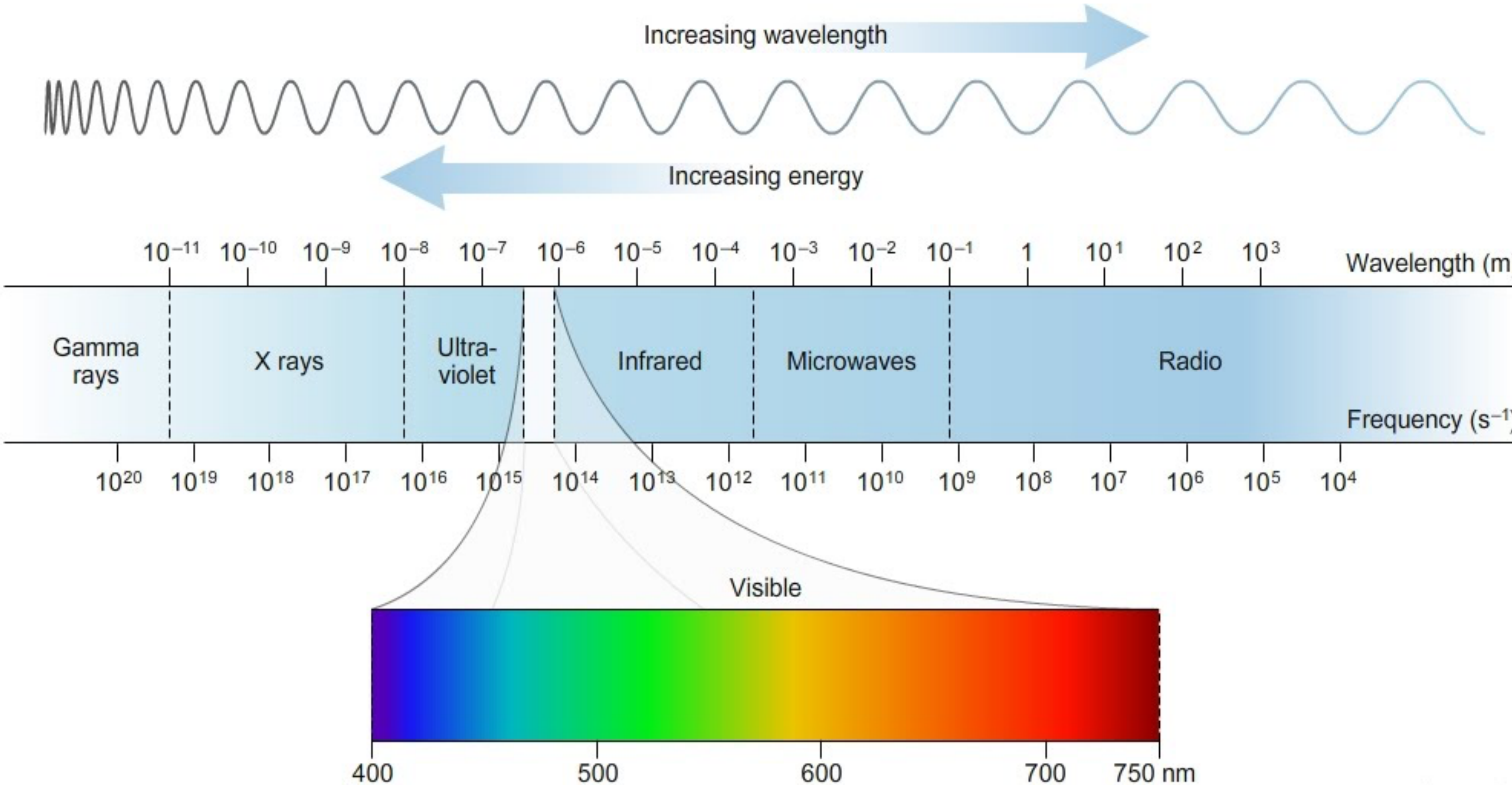
Brightness and flux density

- Brightness is a property of the source – it doesn't change with distance*.
- A bigger telescope doesn't help in seeing lower brightnesses (in fact is often worse!) – the increase in area is offset by the smaller beam
- For extended sources (bigger than the telescope beam) think brightness

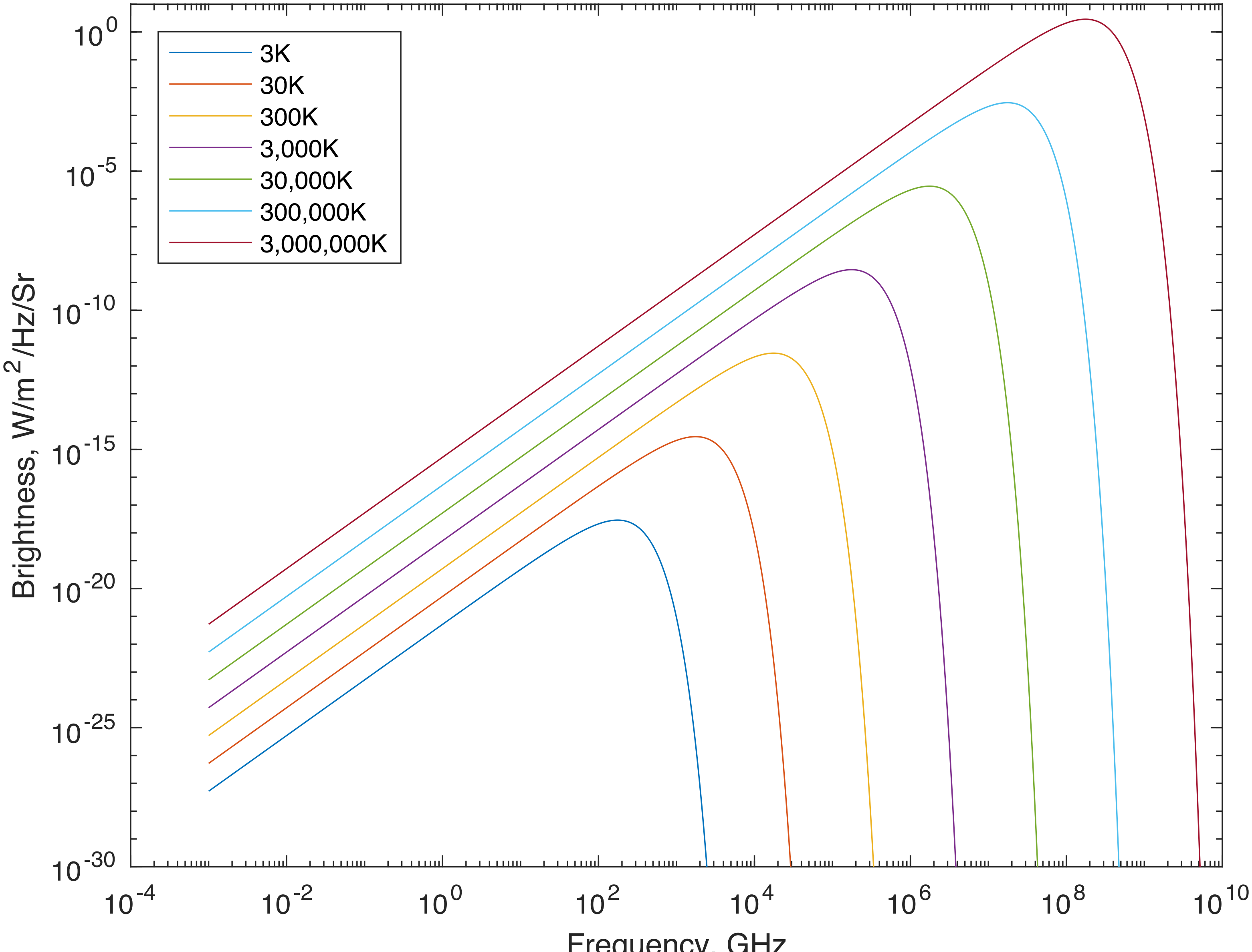
- Flux density depends on the distance to the source
- A bigger telescope (more area) can see lower flux densities
- For point sources (smaller than the telescope beam) think flux density

* Unless you're so far away the universe has expanded noticeably since the light left the source

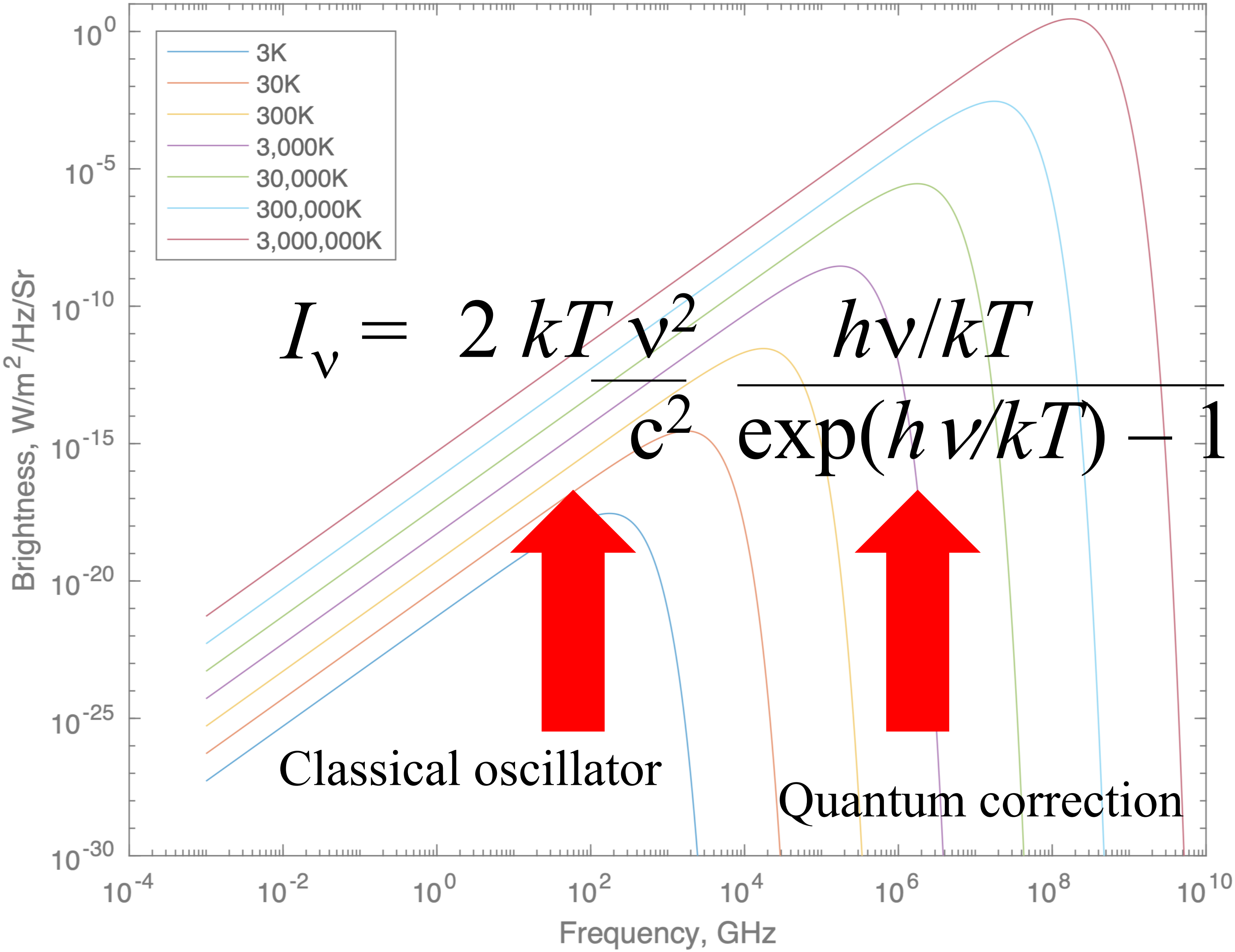
What makes radio astronomy radio?



The Black-body Curve



The Planck Equation

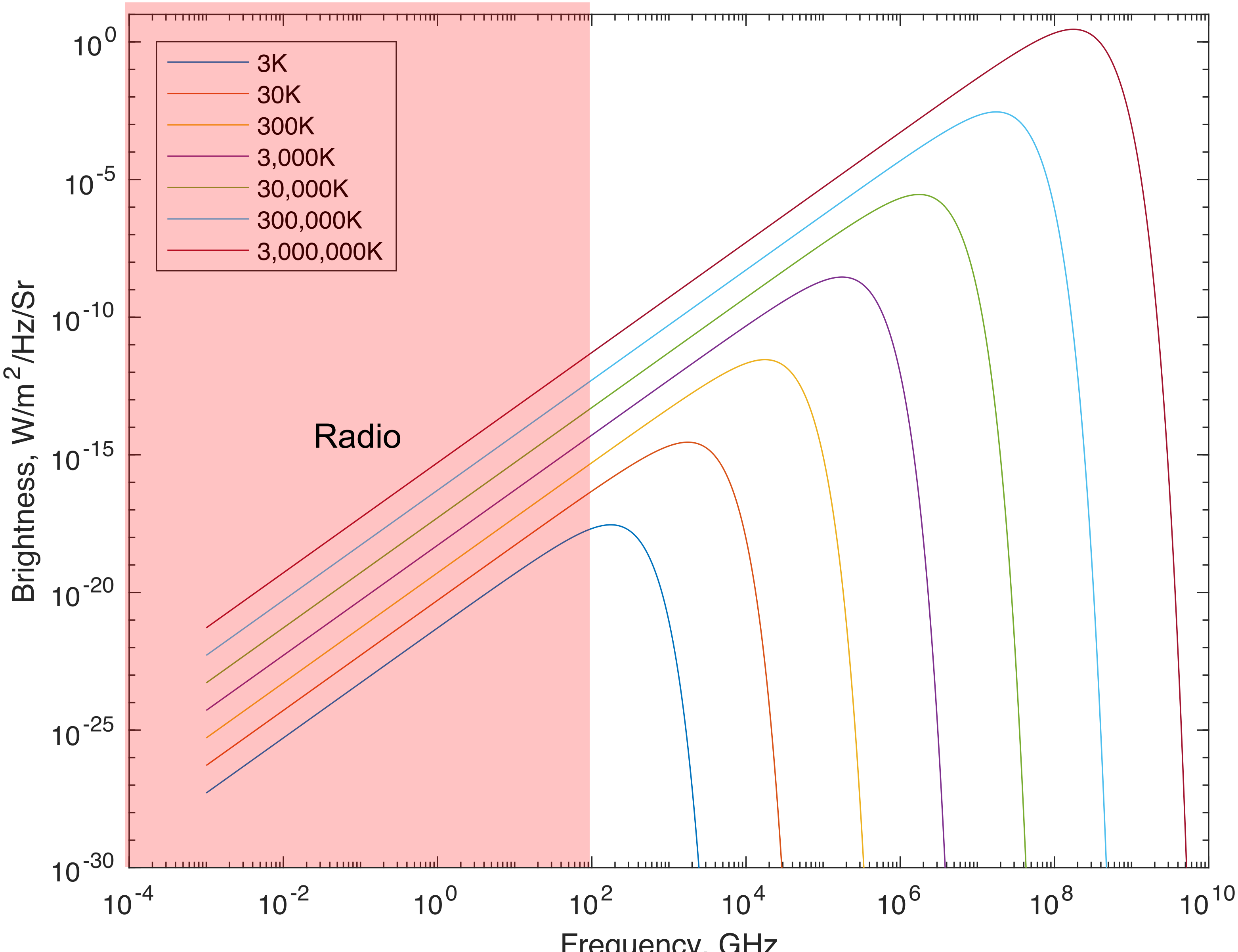


$$I_\nu = 2\,kT\,\frac{\nu^2}{c^2}\;\frac{h\nu/kT}{\exp(h\,\nu/kT) - 1}$$

Classical oscillator

Quantum correction

The Planck Curve



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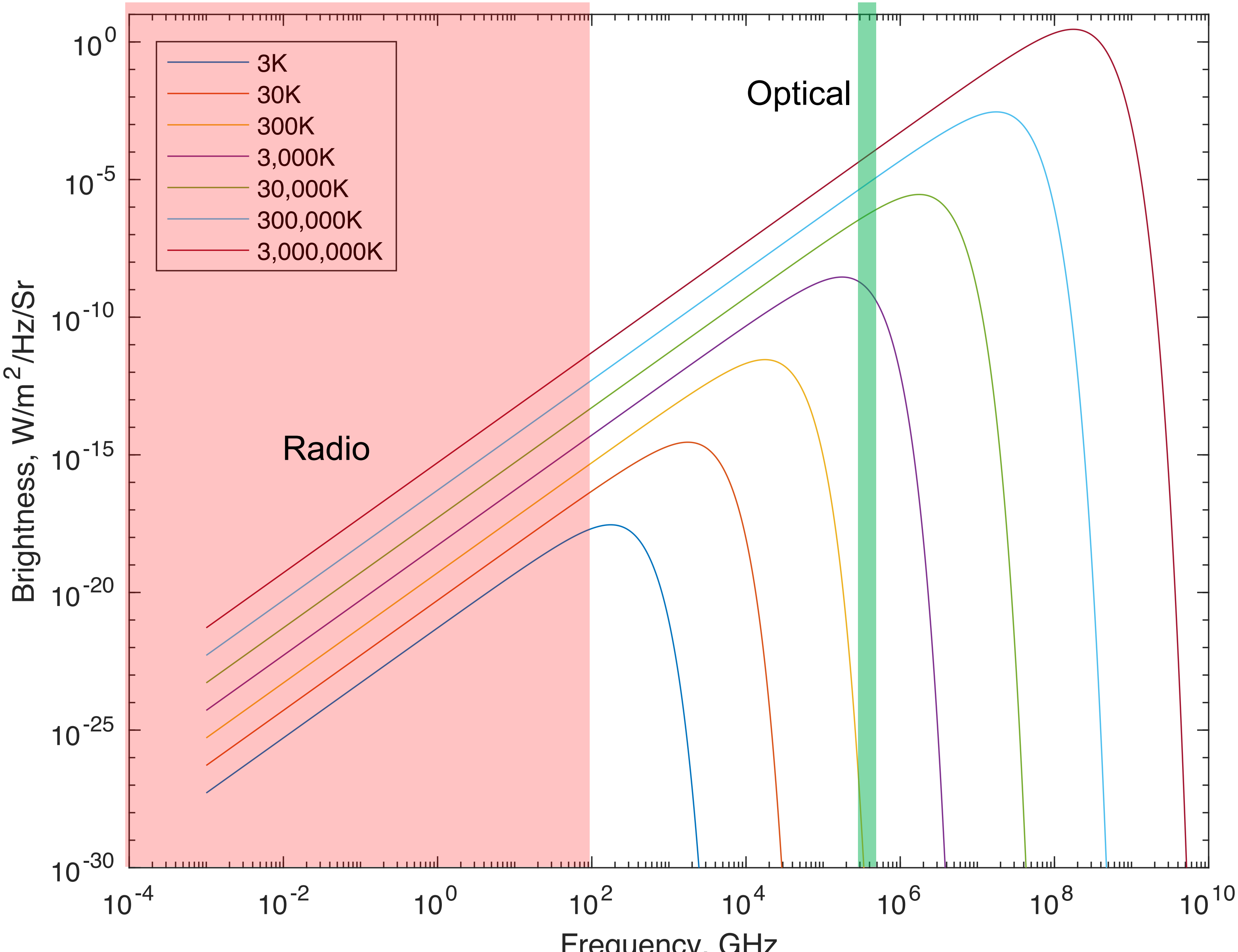

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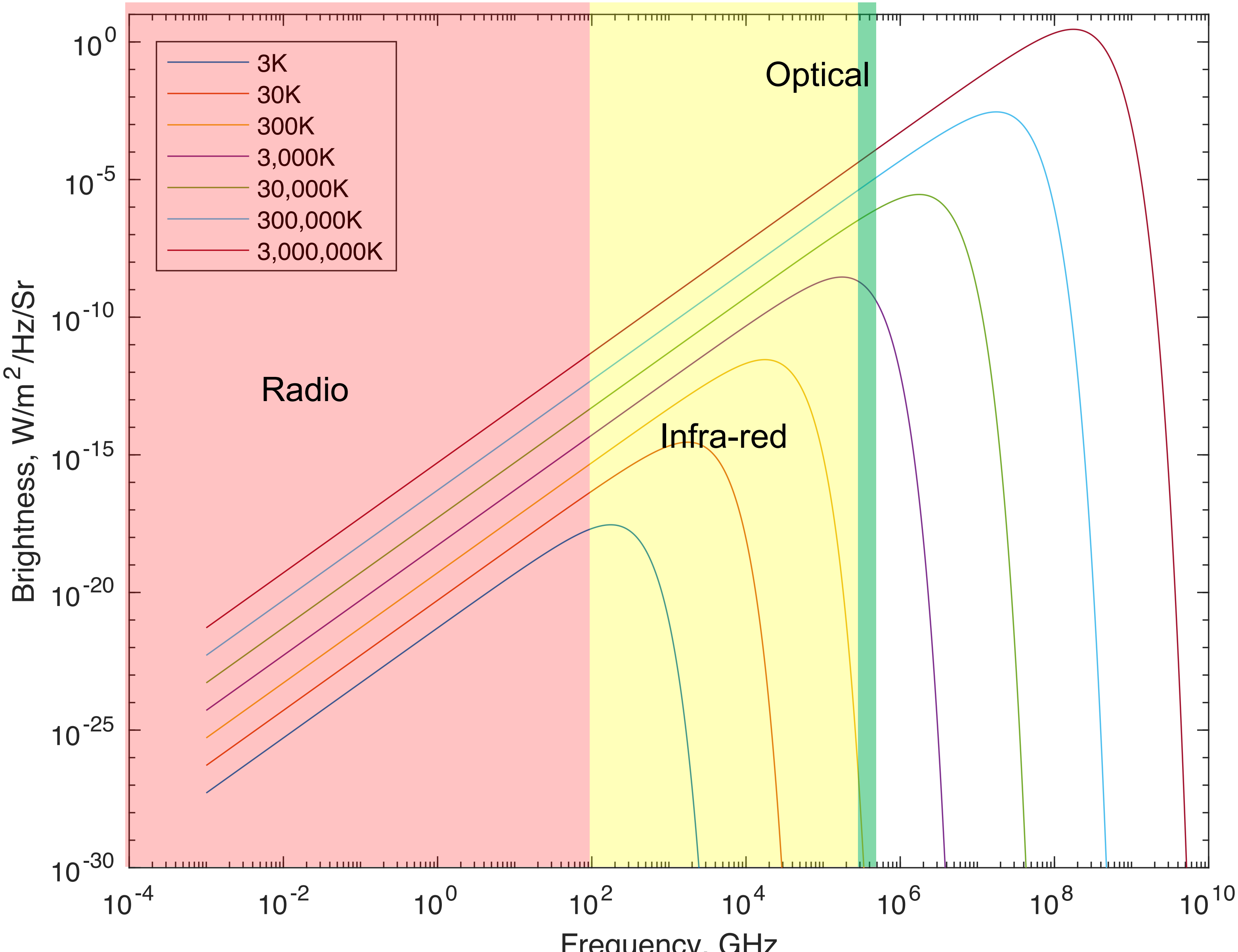

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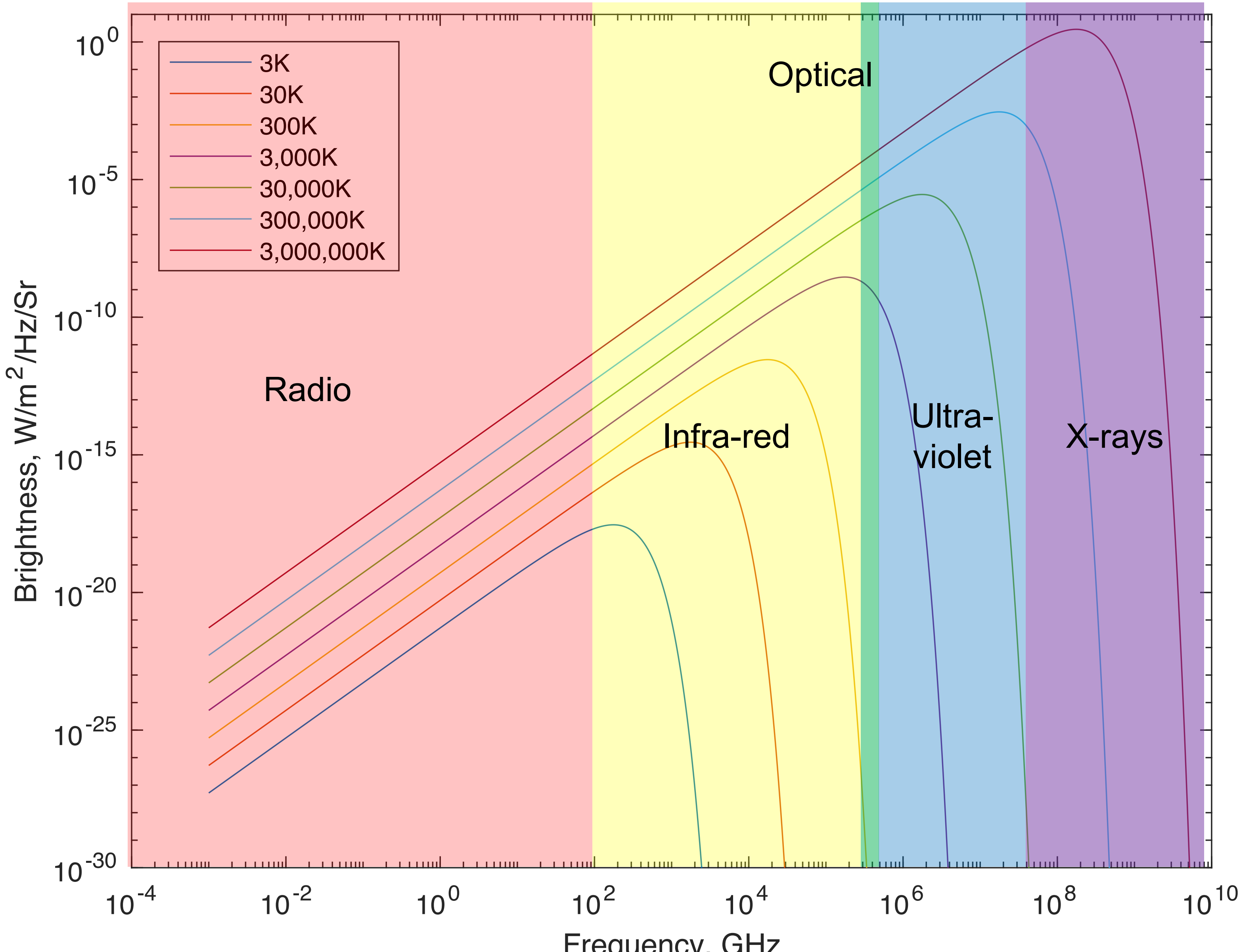


Rayleigh-Jeans brightness temperature

- Measure the brightness I_ν (Jy/sr) of an object at a particular frequency.
- Pretend it is a thermal source so you can use the Rayleigh-Jeans approximation to the Planck curve:

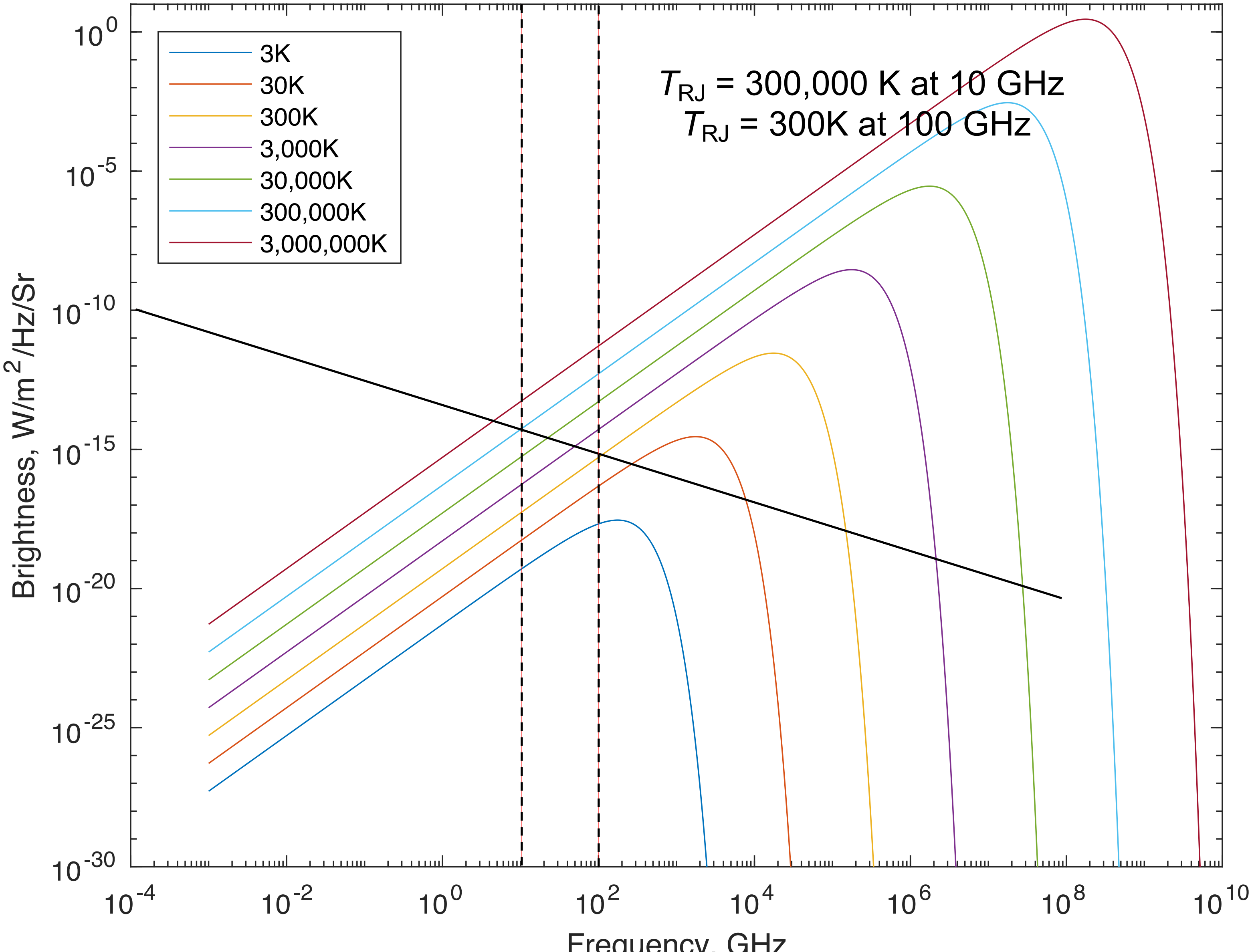
$$I_\nu = \frac{2\,\mathrm{k}T\,\nu^2}{c^2} \frac{h\nu/kT}{\exp(h\nu/kT) - 1}$$

- The brightness temperature is

$$T_{\mathrm{RJ}} = c^2\, I_\nu / 2k\,\nu^2 = I_\nu \lambda^2/2k$$

- This is not a real temperature! (unless the source is a black body)

Rayleigh-Jeans brightness temperature



Rayleigh-Jeans brightness temperature

Now…

$$T_{RJ} = I_\nu \lambda^2/2k$$

and

$$S = \int I_\nu \, d\Omega,$$

$$S = I_\nu \Omega$$

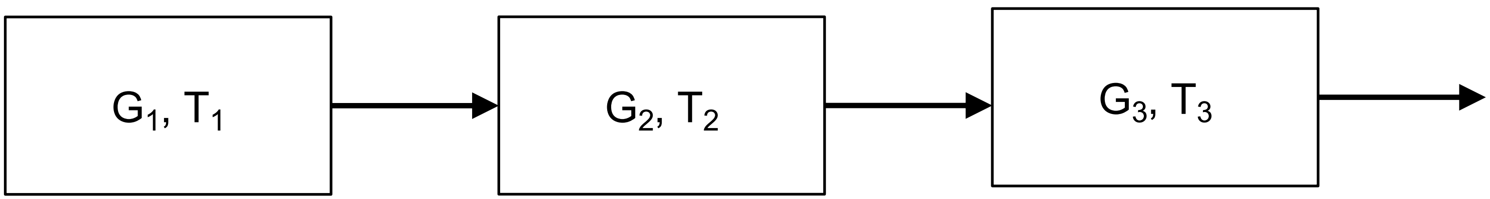
if I_ν is constant over the solid angle, so

$$S / \Omega = 2kT_{RJ} / \lambda^2$$

which links brightness temperature of the source, observed flux density, and the observing wavelength.

Sources of Noise

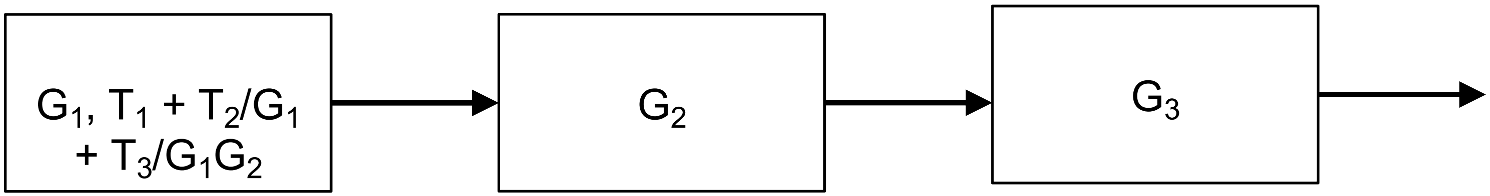
- Everything between the source and the detector has a gain *G* and a noise temperature *T*:



- Each box represents e.g. the atmosphere, the antenna, a component in the signal chain. Apart from amplifiers, everything has a gain $G < 1$.
- For lossy components (including the atmosphere) the noise added is $(1-G)T_{phys}$, so the lossier the component, the more noise it adds.

Sources of Noise

- Imagine moving every source of noise to the beginning of the signal chain – its noise is divided by the gain up to that point:



- So the total noise in the system, referred to the input, is

$$T_{sys} = T_1 + T_2/G_1 + T_3/G_1G_2 + \ldots$$

- This is the *Frijs equation* – components with $G < 1$ add noise, *and* make the noise contribution of the next component bigger
- Eventually you have an amplifier with $G >> 1$ and after that losses don't matter.

Oxford Physics

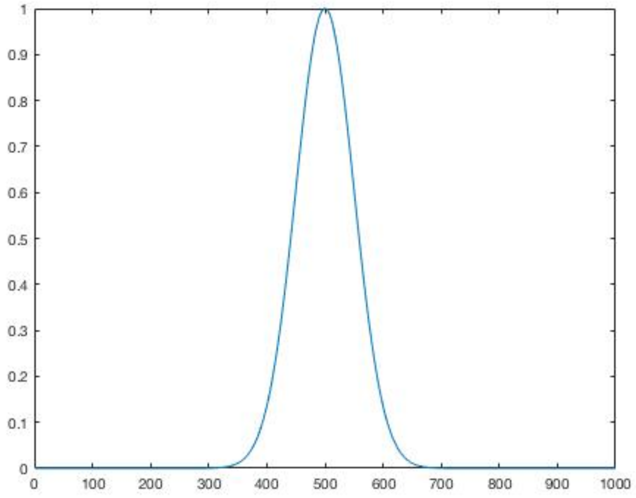
Sensitivity

- We know how to express our astronomical signal as a temperature:

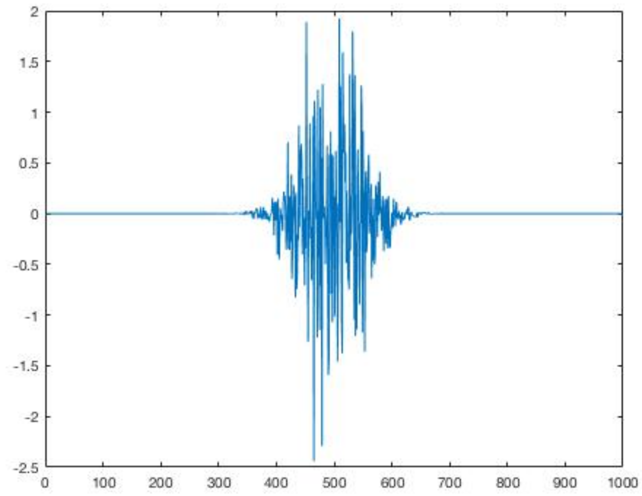
$$T_a = SA_{eff}/2k$$

- We also have our noise expressed as a temperature, T_{sys}.
- What is our sensitivity, i.e. signal-to-noise ratio?
- Signal adds up linearly with number of samples n
- Noise is random, and adds up as $\sqrt{n}$
- So signal-to-noise improves as $\sqrt{n}$

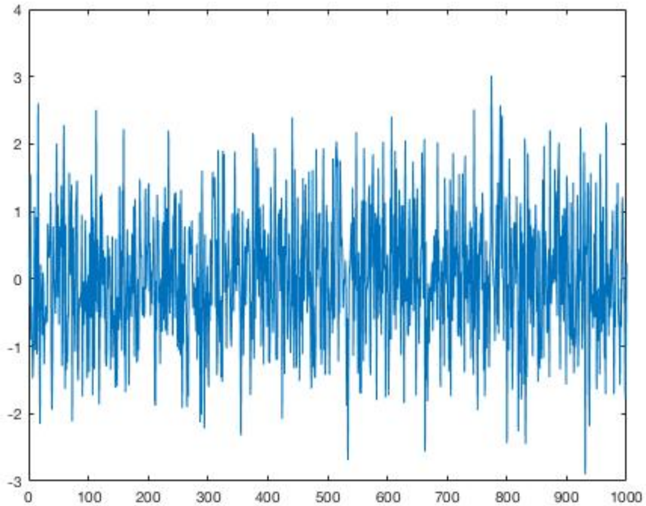
The signal and the noise



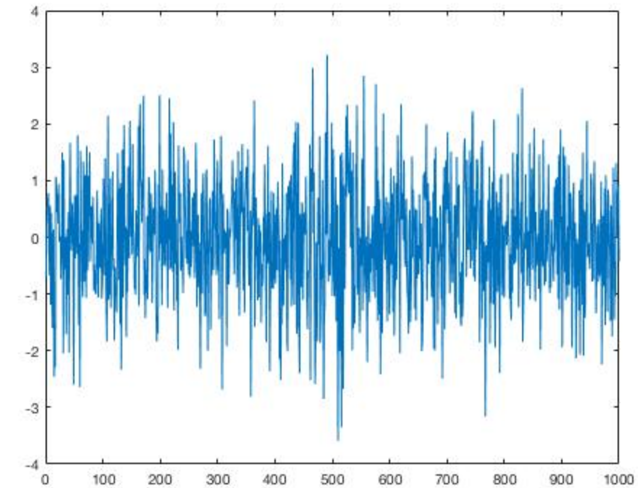
True signal



Signal Voltage

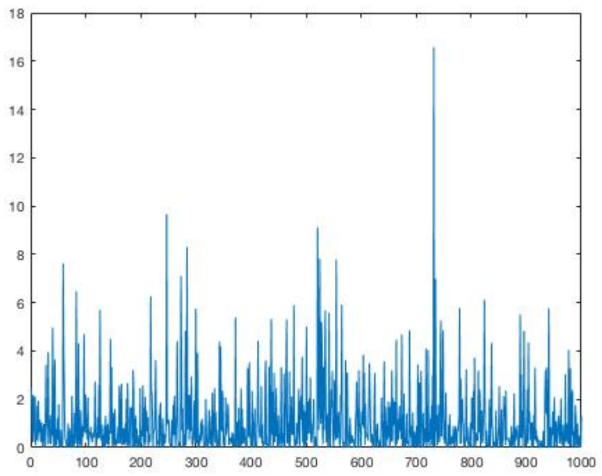


Noise voltage

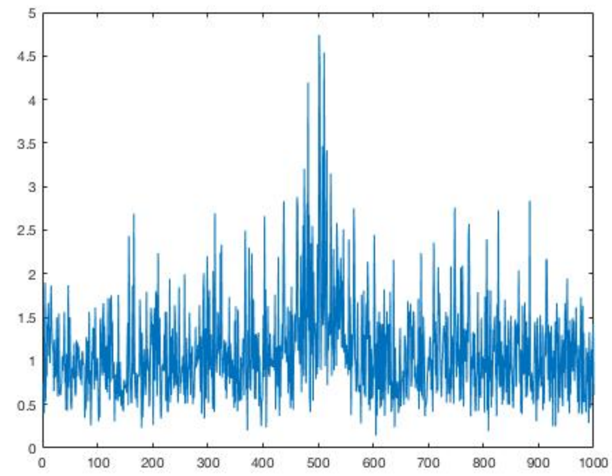


Signal + noise voltage

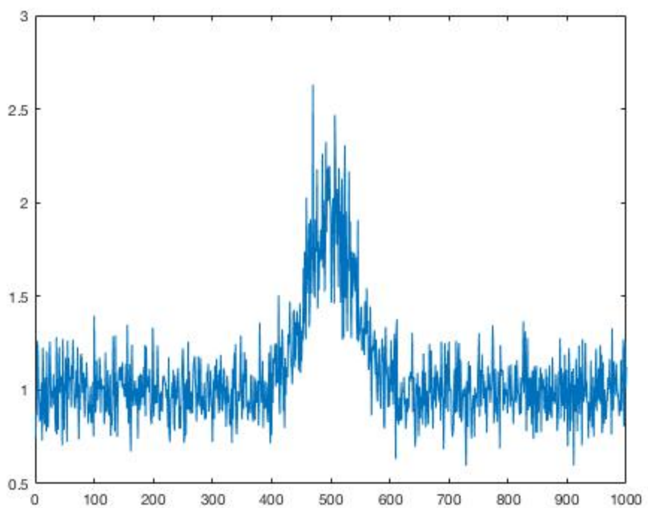
The signal and the noise



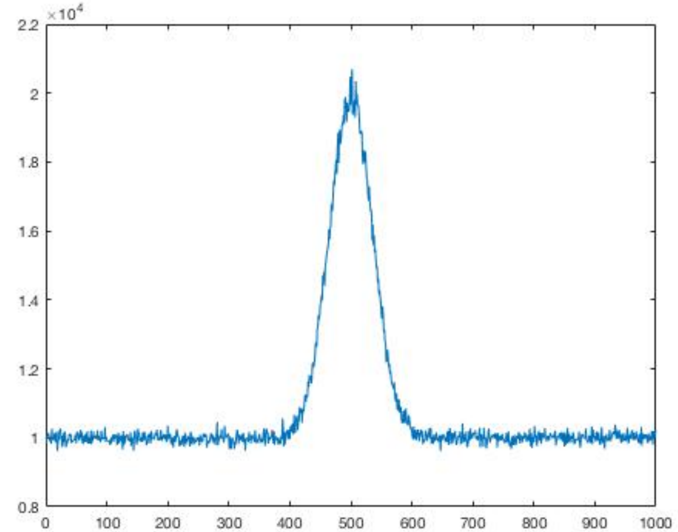
Signal plus noise squared



Averaged 10 times



Averaged 100 times



Averaged 10,000 times

The radiometer equation (1)

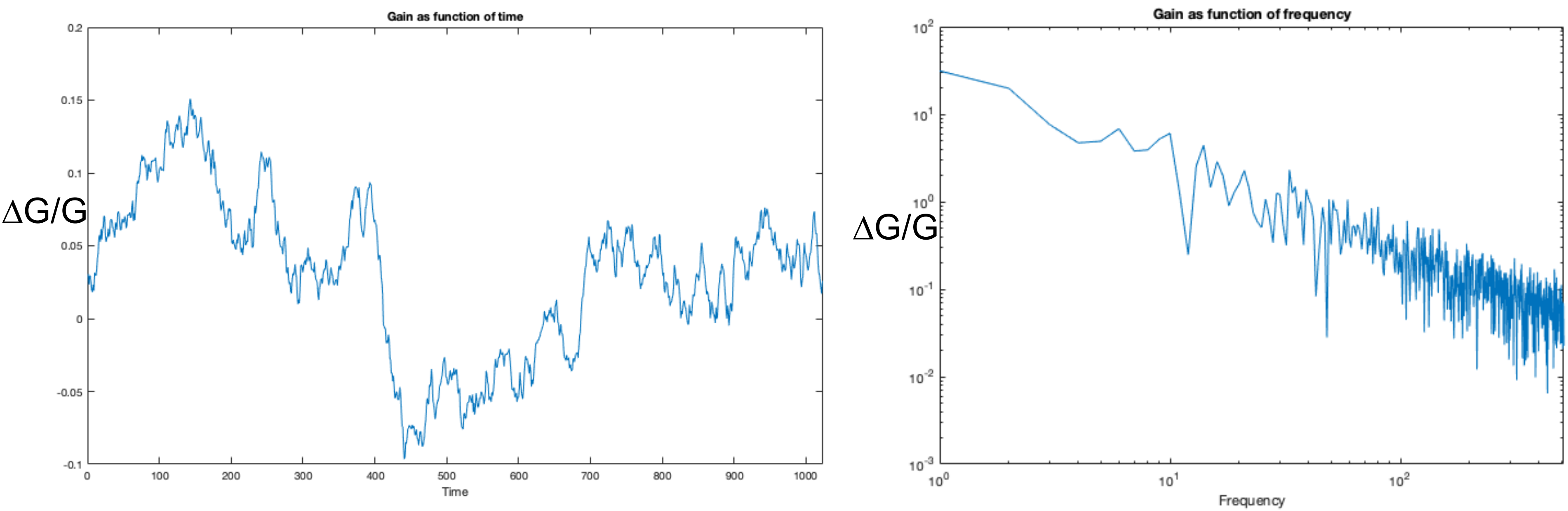
- If the noise has a temperature T_{sys}, the rms fluctuation after averaging n samples is $\sqrt{2}T_{sys}/\sqrt{n}$
- If the signal has a bandwidth Δν, and you observe for time τ, there are 2Δντ independent samples (Nyquist's theorem)
- So the noise rms is $\sigma_T = T_{sys} / \sqrt{\Delta\nu\tau}$
- So a signal-to-noise of one is when

$$T_a = SA_{eff}/2k = \sigma_T = T_{sys} / \sqrt{\Delta\nu\tau}$$

$$\boxed{\begin{aligned}\sigma_T &= T_{sys} / \sqrt{\Delta\nu\tau} \\ \sigma_S &= 2k\, T_{sys} / A_{eff}\sqrt{\Delta\nu\tau}\end{aligned}}$$

1/*f* noise and gain fluctuations

- If your radiometer is perfectly stable, this is the result.
- Most real systems suffer from *gain instability*: the gain wanders around the average value *G* by an amount *ΔG*. *ΔG* is a random variable with a power spectrum that typically varies like $1/f$ (or f^{α} where α can be -0.5 to -2)

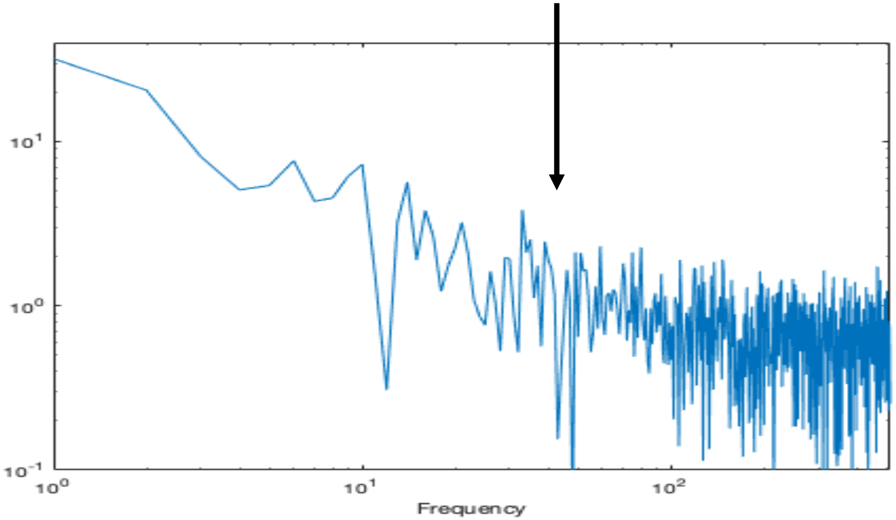


The radiometer equation (2)

- The gain fluctuations add a noise term $T_{sys}.\Delta G/G$ which adds in quadrature to the white noise:

$$\sigma_T = T_{sys} \cdot \sqrt{\frac{1}{\Delta\nu\tau} + \left(\frac{\Delta G}{G}\right)^2}$$

- This means that the noise power spectrum is the sum of the 1/f and white noise components. The frequency at which they are equal is the *knee frequency:*



Summary

- Radio is the part of the E-M spectrum dominated by wave rather than photon behaviour.
- Antennas collect radiation from a region of sky defined by their aperture in wavelengths
- Flux density = power per square metre per Hz of bandwidth
- Brightness = Flux density per solid angle
- Brightness temperature: pretend the brightness you observe is from a Rayleigh-Jeans spectrum (even if it isn't)
- Noise arises from everything between the telescope and the source, plus the antenna and electronics. Losses before the first amplifier matter a lot!
- White noise averages down as the square root of the number of data samples (bandwidth x time).
- Gain instabilities can limit sensitivity at low frequency=long integration times.

Exercises

Have a go at these calculations. You may need to make reasonable estimates of some quantities not given!

1. Antenna temperature

   You are observing the radio source Cassiopeia A (flux density S = 2000 Jy) with a 15-m aperture telescope at 1.4 GHz. What is the antenna temperature?

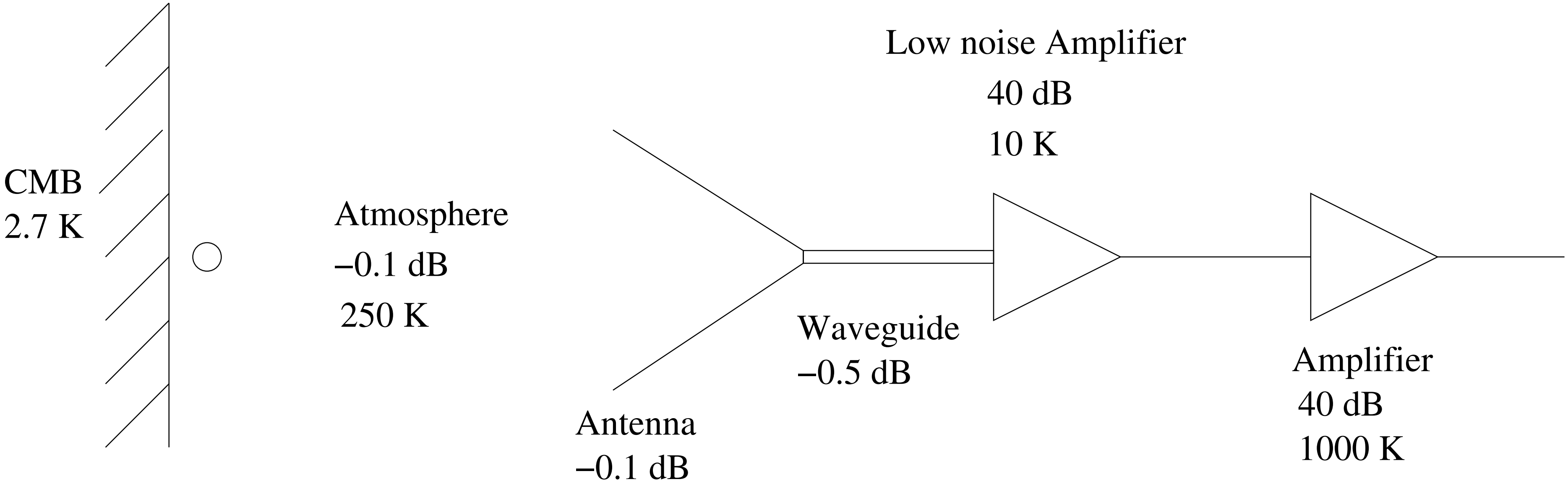
2. Brightness temperature

   Now calculate the beam size of the antenna and use it to calculate the R-J brightness temperature of CasA in this observation

Exercises

3. System temperature

Here is a block diagram of the receiver system of the telescope. Use it to estimate the system temperature.



Exercises

4. Finally, calculate the signal-to-noise ratio for an observation with a bandwidth of 100 MHz and an integration time of 10s.